Stingray Pattern of Dominant Weights

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Goal of the talk

- Partitions with at most r parts $\xrightarrow{\Omega}$ dominant weights of $\mathfrak{sl}_r$.
- Fix integers r, e, w and consider

$$W_{r,e,w} = \Omega\big(\mathcal{P}_{r,e,w}\big) \subset X^+(\mathfrak{sl}_r), \tag{1}$$

  where $\mathcal{P}_{r,e,w}$ is the set of partitions of e–weight w with at most r parts.
- Question: *What does $W_{r,e,w}$ look like as a lattice subset of the dominant chamber?*

Motivation from representation theory

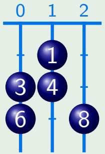
- Blocks of Hecke algebras of type A are classified by e–cores and e–weights.
- The decomposition numbers $d_{\lambda,\mu}$ of Hecke algebras are indexed by partitions in the same block.
- Parabolic Kazhdan–Lusztig polynomials $\mathfrak{n}_{x,y}$ are indexed by dominant weights.
- One has $d_{\lambda,\mu} = \mathfrak{n}_{\Omega(\lambda),\Omega(\mu)}$.

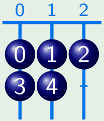
e–cores and e–weight

Let $e \geq 2$ and $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition. We associate to λ an e–abacus with e runners and r beads.

Example

Let $\lambda = (4, 3, 2, 2, 1)$ and $e = 3$. The corresponding e–abacus with 5 beads is





- ***e*–core.** A partition is an e–core if, on its e–abacus, no bead can be moved upward along its runner.
- ***e*–weight.** The e–weight $w_e(\lambda)$ is the total number of upward bead moves required to reach the e–core.

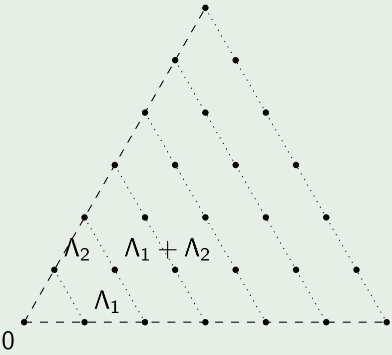
Dominant weights in $\mathfrak{sl}_r$

$\mathfrak{sl}_r$ is the Lie algebra of $r \times r$ complex matrices with trace zero, with Lie bracket $[x, y] = xy - yx$.

The weight lattice of $\mathfrak{sl}_r$ has basis $\Lambda_1, \ldots, \Lambda_{r-1}$, the fundamental weights. A weight is dominant if all coefficients in this basis are non–negative.

Example

For $\mathfrak{sl}_3$, the dominant weights form the lattice $X^+(\mathfrak{sl}_3)$ shown below:



From partitions to dominant weights

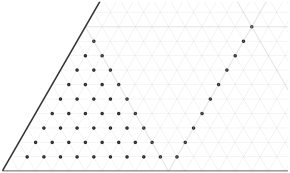
- Fix $r \geq 2$. Write $\mathcal{P}[r]$ for partitions with at most r parts.
- For $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}[r]$, the corresponding dominant weight of $\mathfrak{sl}_r$ is

$$\Omega(\lambda) = \sum_{i=1}^{r-1} (\lambda_i - \lambda_{i+1} + 1)\,\Lambda_i. \tag{2}$$

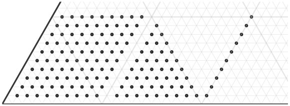
- Fix $e \geq 2$ and $w \geq 0$. Let $\mathcal{P}_{r,e,w}$ be the set of partitions with at most r parts and e–weight w.
- Define

$$W_{r,e,w} = \Omega\big(\mathcal{P}_{r,e,w}\big) \subset X^+(\mathfrak{sl}_r). \tag{3}$$

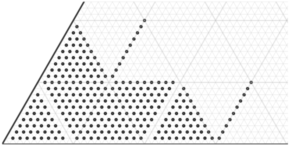
Example: $W_{3,10,w}$



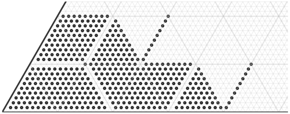
$(r, e, w) = (3, 10, 0)$



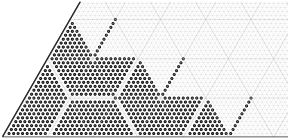
$(r, e, w) = (3, 10, 1)$



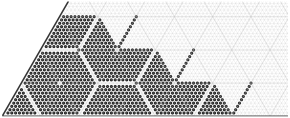
$(r, e, w) = (3, 10, 2)$



$(r, e, w) = (3, 10, 3)$

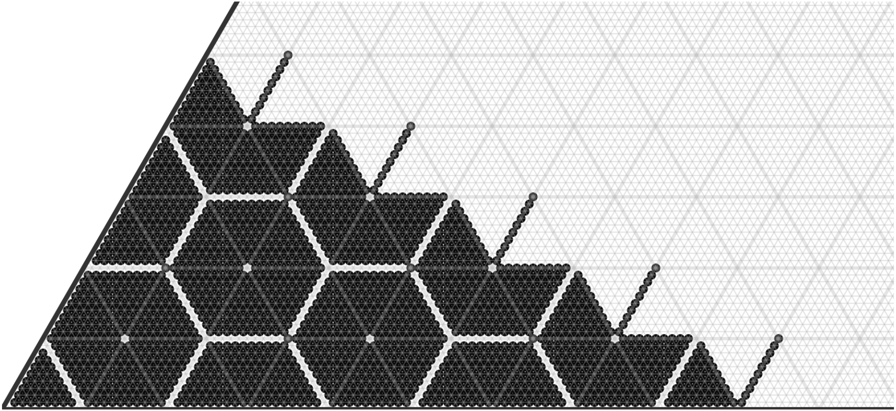


$(r, e, w) = (3, 10, 4)$



$(r, e, w) = (3, 10, 5)$

Example: $W_{3,10,8}$



Case $w = 0$: simplices

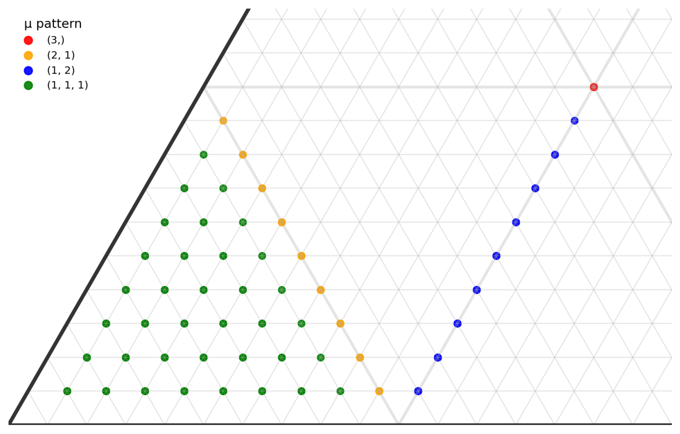
Theorem (Simplicial structure for e–cores)

Fix integers $e > r > 0$. Then $W_{r,e,0} = \bigcup_{\mu \models r} W_{r,e,0}(\mu)$. If μ is a composition of r of length j, then $W_{r,e,0}(\mu)$ is the set of lattice points of a $(j-1)$–dimensional simplex of side length $e - j$.

Case $w = 0$: simplices

Example

When $(r, e, w) = (3, 10, 0)$, we have
$W_{r,e,0} = W_{r,e,0}(1,1,1) \cup W_{r,e,0}(2,1) \cup W_{r,e,0}(1,2) \cup W_{r,e,0}(3)$, and these sets look as follows:

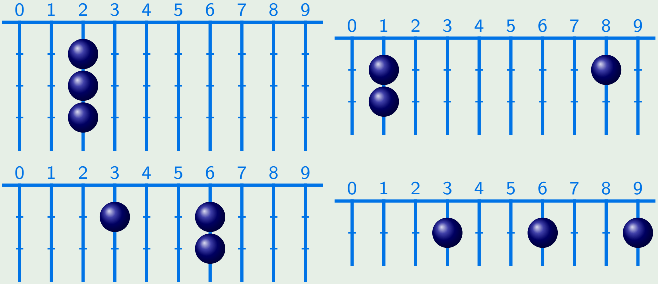


$W_{r,e,w}(\mu)$

The composition $\mu = (\mu_1, \ldots, \mu_j)$ corresponds to placing μ_i beads on the i–th runner out of the chosen j runners.

Example

For $(r, e, w) = (3, 10, 0)$, the sets $W_{r,e,0}(3)$, $W_{r,e,0}(2, 1)$, $W_{r,e,0}(1, 2)$ and $W_{r,e,0}(1, 1, 1)$ correspond to the following abaci:



Case $w = 0$

Corollary (lattice points in one simplex)

If μ has length j, then

$$\#W_{r,e,0}(\mu) = \binom{e-1}{j-1}. \tag{4}$$

Corollary (total size of $W_{r,e,0}$)

If $r < e$, then

$$|W_{r,e,0}| = \binom{e+r-2}{r-1}. \tag{5}$$

- For each j, there are $\binom{r-1}{j-1}$ compositions of r of length j.
- Summing

$$\sum_{j=1}^{r} \binom{r-1}{j-1}\binom{e-1}{j-1}$$

and applying Vandermonde identity.

$w > 0$: reusing the same simplices

Set-up for general w

For each composition $\mu = (\mu_1, \ldots, \mu_j)$ of r, define $W_{r,e,w}(\mu)$ as the subset of $W_{r,e,w}$ coming from e–cores of type μ and e–weight w.

Theorem (Simplicial decomposition for all w)

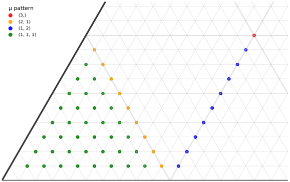
Fix $e > r$ and $w \in \mathbb{N}$.

(a) The sets $W_{r,e,w}(\mu)$ for distinct μ are pairwise disjoint.

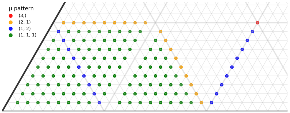
(b) Each $W_{r,e,w}(\mu)$ is a disjoint union of copies of $W_{r,e,0}(\mu)$.

(c) The number of copies equals $A(\mu; w)$, the number of j–partitions of w of type μ.

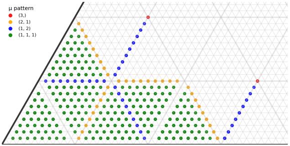
Example: $W_{3,e,w}$



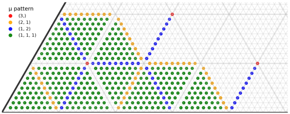
$(r, e, w) = (3, 10, 0)$



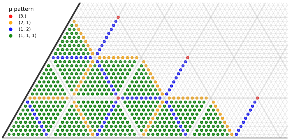
$(r, e, w) = (3, 10, 1)$



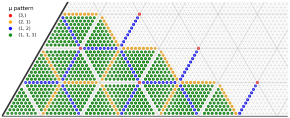
$(r, e, w) = (3, 10, 2)$



$(r, e, w) = (3, 10, 3)$

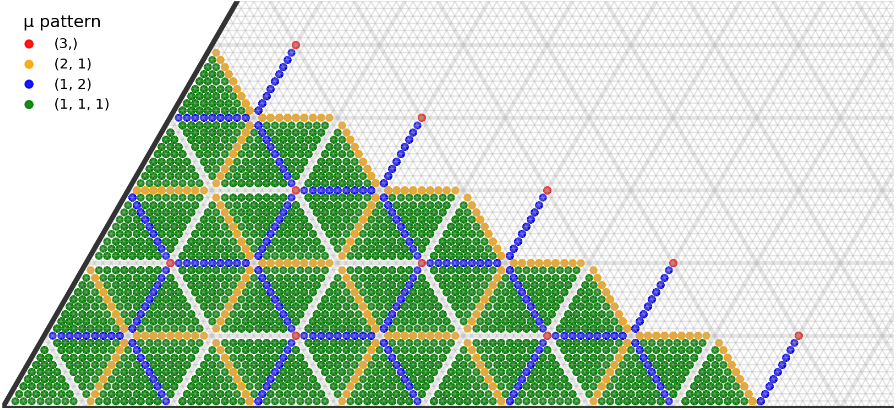


$(r, e, w) = (3, 10, 4)$



$(r, e, w) = (3, 10, 5)$

Example: $W_{3,10,8}$

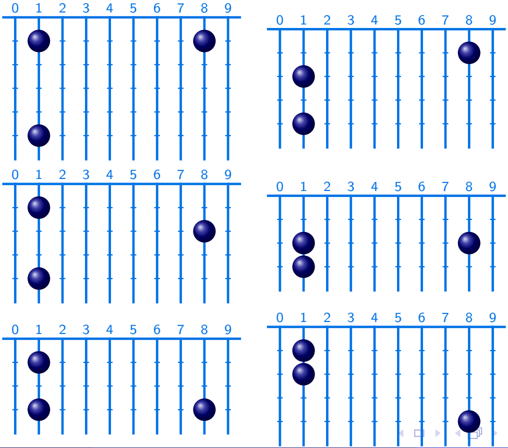


Abaci translation

Let $(r, e, w) = (3, 10, 3)$ and consider $W_{r,e,w}(2, 1)$. The set of 2–partitions of $w = 3$ of type $(2, 1)$ is

$$\lambda_1 = (3|\emptyset),\ \lambda_2 = (2,1|\emptyset),\ \lambda_3 = (2|1),\ \lambda_4 = (1,1|1),\ \lambda_5 = (1|2),\ \lambda_6 = (\emptyset|3).$$

They correspond to the following abaci:



Counting formula for general w

For each composition μ of r with length j:

- There are $A(\mu; w)$ copies of the simplex $W_{r,e,0}(\mu)$ inside $W_{r,e,w}(\mu)$.
- Each copy contributes $\binom{e-1}{j-1}$ lattice points.

Hence

$$|W_{r,e,w}(\mu)| = A(\mu; w)\binom{e-1}{j-1}, \tag{6}$$

and summing over all composition types $\mu \models r$ yields

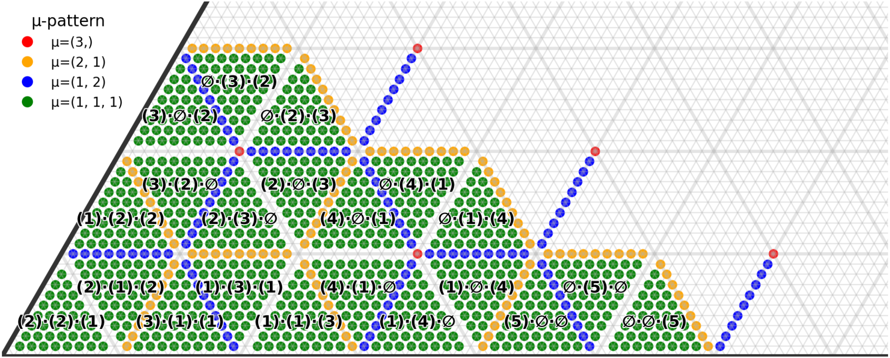
$$|W_{r,e,w}| = \sum_{\mu \models r} A(\mu; w)\binom{e-1}{\ell(\mu)-1}. \tag{7}$$

- For $r = 3$, this can be simplified further to a quadratic expression in e with explicit coefficients depending on w.

Indexing e–alcoves by weak compositions

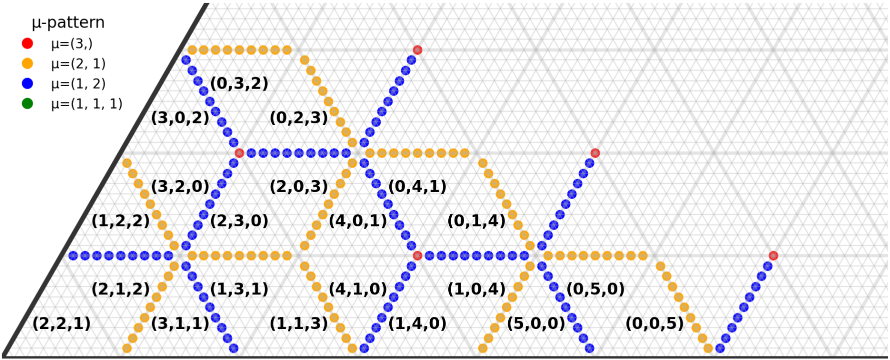
- Using the labels arising from $W_{r,e,w}(1^r)$ (beads on all r runners), one obtains a combinatorial labelling of certain bounded families of e–alcoves by j–partitions of w of type (1^r).
- A j–partition $(\lambda_1|\lambda_2|\cdots|\lambda_j)$ of type (1^r) can be identified with the weak composition $(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_j|)$.

$(r, e, w) = (3, 10, 5)$



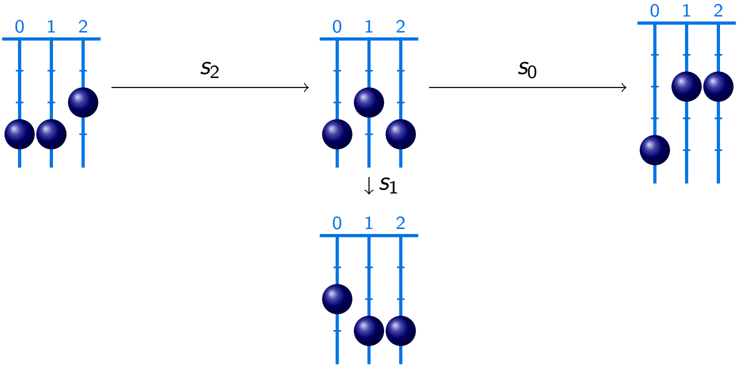
μ-pattern
μ=(3,)
μ=(2, 1)
μ=(1, 2)
μ=(1, 1, 1)
∅·(3)·(2)
(3)·∅·(2)
∅·(2)·(3)
(3)·(2)·∅
(2)·∅·(3)
∅·(4)·(1)
(1)·(2)·(2)
(2)·(3)·∅
(4)·∅·(1)
∅·(1)·(4)
(2)·(1)·(2)
(1)·(3)·(1)
(4)·(1)·∅
(1)·∅·(4)
∅·(5)·∅
(2)·(2)·(1)
(3)·(1)·(1)
(1)·(1)·(3)
(1)·(4)·∅
(5)·∅·∅
∅·∅·(5)

$(r, e, w) = (3, 10, 5)$


μ-pattern
μ=(3,)
μ=(2, 1)
μ=(1, 2)
μ=(1, 1, 1)
(0,3,2)
(3,0,2)
(0,2,3)
(3,2,0)
(2,0,3)
(0,4,1)
(1,2,2)
(2,3,0)
(4,0,1)
(0,1,4)
(2,1,2)
(1,3,1)
(4,1,0)
(1,0,4)
(0,5,0)
(2,2,1)
(3,1,1)
(1,1,3)
(1,4,0)
(5,0,0)
(0,0,5)

Affine Weyl group action

For $\mathfrak{sl}_r$, the corresponding affine Weyl group has generators $s_0, s_1, \ldots, s_{r-1}$. It acts on the e–alcoves by reflections, and it also acts on the abacus:



Thanks!